A bialgebraic semantics for natural numbers in positional notation

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A language $\mathcal{S}$ for natural numbers in positional notation

The syntax of $\mathcal{S}$ is the following:

$$(\mathcal{S} \ni)\ S ::= \mathsf{nil} \mid \langle \mathbf{d}, S \rangle \mid S\ ;\ S$$
$$\mathbf{d} ::= 0 \mid \ldots \mid 9$$

where

- nil denotes the empty list;
- $\langle \mathbf{d}, - \rangle$ is the cons operator;
- ; appends two lists.

The language can be seen as a Σ-algebra for the following functor $\Sigma : \mathit{Set} \to \mathit{Set}$

$$\Sigma(X) = 1 + \mathbf{D} \times X + X \times X \qquad \Sigma(f) = id_1 + id_{\mathbf{D}} \times f + f \times f$$

where $\mathbf{D} = 1+1+1+1+1+1+1+1+1+1$ is the constant algebra modeling digits.

Denotational semantics of $\mathcal{S}$ - I

Let $\mathbb{N}^\omega \equiv \mathbb{N} \cup \{\omega\}$.
Then consider $\mathbb{N}^\omega_\perp \equiv \mathbb{N}^\omega \cup \{\perp\}$, which can be endowed naturally with the structure of a Σ-algebra:

$$\iota : 1 + \mathbf{D} \times \mathbb{N}^\omega_\perp + \mathbb{N}^\omega_\perp \times \mathbb{N}^\omega_\perp \to \mathbb{N}^\omega_\perp$$

where ι is defined as

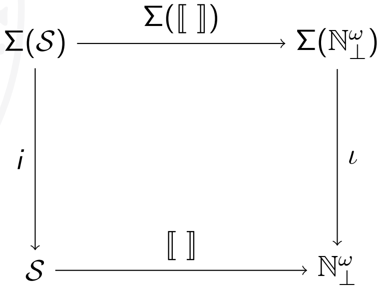
$$\iota = (\lambda x.\, \perp) \oplus \left(\lambda d\, x. \begin{cases} d & \text{if } x = \perp \\ \omega & \text{if } x = \omega \\ d \cdot 10^{\lceil log_{10} x \rceil} + x & \text{otherwise} \end{cases} \right) \oplus \left(\lambda x_1\, x_2. \begin{cases} x_2 & \text{if } x_1 = \perp \\ x_1 & \text{if } x_2 = \perp \\ \omega & \text{if } x_1 = \omega \text{ or } x_2 = \omega \\ x_1 \cdot 10^{\lceil log_{10} x_2 \rceil} + x_2 & \text{otherwise} \end{cases} \right)$$

Denotational semantics of $\mathcal{S}$ - II

Let $\mathcal{S} = \mu X.\Sigma(X)$, that is the least fix point of the operator T:

$$T_0 = \emptyset$$
$$T_{n+1} = \{\mathrm{nil}\} \cup \{\langle \mathbf{d}, s\rangle \mid \mathbf{d} \in \mathbf{D} \wedge s \in T_n\} \cup \{s_1 \,;\, s_2 \mid s_1, s_2 \in T_n\}\,.$$

Denotational semantics is an initial semantics, i.e., $[\![\ ]\!]$ is the unique Σ-algebra morphism, called *interpretation* between the initial (i.e., free) Σ-algebra $(\mathcal{S}, i)$ and $(\mathbb{N}^{\omega}_{\perp})$



Denotational semantics of $\mathcal{S}$ - III

The interpretation is defined as

$$[\![\ ]\!] : \mathcal{S} \to \mathbb{N}_\perp^\omega$$

$$[\![\text{nil}]\!] = \perp$$

$$[\![\langle \mathbf{d}, s\rangle]\!] = \begin{cases} \mathbf{d} & \text{if } [\![s]\!] = \perp \\ \omega & \text{if } [\![s]\!] = \omega \\ \mathbf{d} \cdot 10^{\lceil log_{10} x\rceil} + [\![s]\!] & \text{otherwise} \end{cases}$$

$$[\![s_1 \ ; \ s_2]\!] = \begin{cases} [\![s_2]\!] & \text{if } [\![s_1]\!] = \perp \\ [\![s_1]\!] & \text{if } [\![s_2]\!] = \perp \\ \omega & \text{if } [\![s_1]\!] = \omega \text{ or } [\![s_2]\!] = \omega \\ [\![s_1]\!] \cdot 10^{\lceil log_{10} [\![s_2]\!]\rceil} + [\![s_2]\!] & \text{otherwise} \end{cases}$$

Transition semantics of $\mathcal{S}$ - I

Expressions in $\mathcal{S}$ can be viewed also as agents or systems, possibly interacting with the environment by producing some action or sending some message.

Hence, in natural deduction style, we can describe the behaviour of terms using a labelled transition system.

Denote by $\mathcal{R}$ the following set of rules (in tyft-format).

$$\frac{}{\text{nil} \xrightarrow{\surd} \text{nil}} \qquad \frac{}{\langle \mathbf{d}, s \rangle \xrightarrow{\mathbf{d}} s}$$

$$\frac{s \xrightarrow{\mathbf{d}} s'}{s\,;\,t \xrightarrow{\mathbf{d}} s'\,;\,t} \qquad \frac{s \xrightarrow{\surd} s' \quad t \xrightarrow{\ell} t' \quad \ell \in \{\surd, \mathbf{d}\}}{s\,;\,t \xrightarrow{\ell} t'}$$

The natural deduction system describes a small step semantics.

Transition semantics of $\mathcal{S}$ - II

Transition semantics is an operational semantics, that is, it defines a behavior. The appropriate categorical view is that of a final coalgebraic semantics.

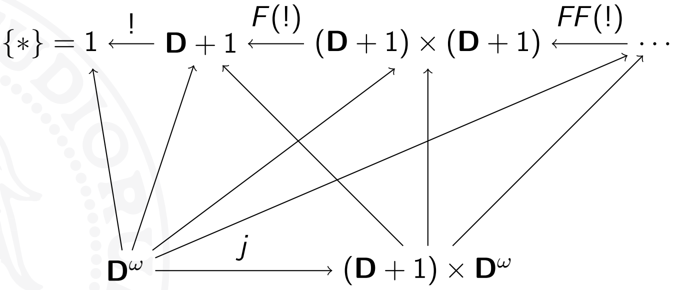
Consider the functor $F : Set \rightarrow Set$ defined as

$$F(X) = (\mathbf{D} + 1) \times X \qquad F(f) = (id_{\mathbf{D}} + id_1) \times f$$

$\mathcal{S}$ can be endowed with a structure of F-coalgebra induced by $\xrightarrow{\ell}$:

$$\alpha(s) = \begin{cases} (\surd, s') & \text{if } s \xrightarrow{\surd} s' \\ (\mathbf{d}, s') & \text{if } s \xrightarrow{\mathbf{d}} s' \end{cases} \qquad \begin{array}{c} \mathcal{S} \\ \alpha \big\downarrow \\ F(\mathcal{S}) \end{array}$$

Construction of the final F-coalgebra



Lemma

The final F-coalgebra is $(\mathbf{D}^\omega, j)$ which is the limit of the diagram above and j is unique mediating arrow from $\mathbf{D}^\omega$ to $F(\mathbf{D}^\omega)$.

Intuitively, $\mathbf{D}^\omega$ is the set of infinite sequences over $\mathbf{D} \cup \{\surd\}$.

Final semantics of $\mathcal{S}$

The final semantics of $\mathcal{S}$ is $(\!|\ |\!)$ such that

$$\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\quad (\!|\ |\!) \quad} & \mathbf{D}^{\omega} \\
\Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle j} \\
F(\mathcal{S}) & \xrightarrow{\quad F((\!|\ |\!)) \quad} & F(\mathbf{D}^{\omega})
\end{array}$$

that is, let . denotes concatenation then

$$j \circ (\!|\ |\!)(s) = \begin{cases} \surd.(\!|s'|\!) & \text{if } s \xrightarrow{\surd} s' \\ \mathbf{d}.(\!|s'|\!) & \text{if } s \xrightarrow{\mathbf{d}} s' \end{cases}$$

Full abstraction

We want to prove that the denotational semantics corresponds to the transition semantics.

Theorem

For all $s, t \in \mathcal{S}$ we have that

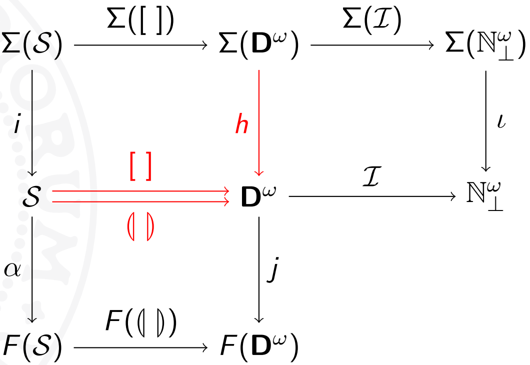
$$[\![s_1]\!] = [\![s_2]\!] \iff (\!| s_1 |\!) = (\!| s_2 |\!)$$

Proof.

Show that a bialgebraic semantics can be given for $\mathcal{S}$. □

Bialgebraic semantics of $\mathcal{S}$

$\mathbf{D}^\omega$ can be endowed with a structure of both an F-coalgebra and a Σ-algebra: (where $[\![\ ]\!] = \mathcal{I} \circ [\ ]$)



Problems to solve

How to define h? and How to prove $[\ ] = (\!|\ |\!)$?

Introducing mixed terms - I

We introduce a set $\mathcal{S}_{\mathbf{D}^\omega}$ of mixed terms, that is, the set of terms over the extended signature $\Sigma + \mathbf{D}^\omega$: the original signature where all the elements of $\mathbf{D}^\omega$ are added as *constants*.

E.g., $\langle \mathbf{d}, s \rangle$ or $\mathrm{nil}\,; s$ are mixed terms, when s in $\mathbf{D}^\omega$.

Notice that, for definition of Σ

$$(\Sigma + \mathbf{D}^\omega)(S) \cong \Sigma(S) + \mathbf{D}^\omega$$

Since $\mathcal{S}_{\mathbf{D}^\omega}$ is the fix point of $(\Sigma + \mathbf{D}^\omega)$, this implies $\mathcal{S}_{\mathbf{D}^\omega} \cong \Sigma(\mathcal{S}_{\mathbf{D}^\omega}) + \mathbf{D}^\omega$ and giving the existence of two functions

$$\begin{array}{ccc} \Sigma(\mathcal{S}_{\mathbf{D}^\omega}) & & \\ \downarrow \kappa & & \\ \mathcal{S}_{\mathbf{D}^\omega} & \xleftarrow{\ \phi\ } & \mathbf{D}^\omega \end{array}$$

ϕ is an injection, which includes infinite sequences into mixed terms

κ is an isomorphism

Introducing mixed terms - II

By initiality of $(\mathcal{S}, i)$ there exists a unique morphism $\varphi : (\mathcal{S}, i) \to (\mathcal{S}_{\mathbf{D}^\omega}, \kappa)$

$$
\begin{array}{ccccc}
\Sigma(\mathcal{S}) & \xrightarrow{\Sigma(\varphi)} & \Sigma(\mathcal{S}_{\mathbf{D}^\omega}) & & \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \kappa} & & \\
\mathcal{S} & \overset{\varphi}{\rightarrowtail} & \mathcal{S}_{\mathbf{D}^\omega} & \overset{\phi}{\leftarrowtail} & \mathbf{D}^\omega
\end{array}
$$

Notice that φ is an injection, which includes the terms of the language $\mathcal{S}$ into the set of mixed terms.

Transition system for mixed terms - I

We can define a transition system for the set of mixed terms as follows (recall that $j : \mathbf{D}^\omega \to F(\mathbf{D}^\omega)$)

$$\mathcal{R}_{\mathbf{D}^\omega} = \mathcal{R} \cup \{p \xrightarrow{\ell} q \mid j(p) = (\ell, q)\} \quad (\ell \in \{\surd, \mathbf{d}\})$$

That is, all possible transition in $(\mathbf{D}^\omega, j)$ are added to $\mathcal{R}$ as axioms.

Hence, $\mathcal{S}_{\mathbf{D}^\omega}$ can be turned into a F-coalgebra.

$$\begin{array}{ccccc}
\mathcal{S} & \xrightarrow{\ \varphi\ } & \mathcal{S}_{\mathbf{D}^\omega} & \xleftarrow{\ \phi\ } & \mathbf{D}^\omega \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle j} \\
F(\mathcal{S}) & \xrightarrow{\ F(\varphi)\ } & F(\mathcal{S}_{\mathbf{D}^\omega}) & \xleftarrow{\ F(\phi)\ } & F(\mathbf{D}^\omega)
\end{array}$$

Where β is defined as $\beta(t) = (\ell, t') \iff t \xrightarrow{\ell}_{\mathbf{D}^\omega} t'$.

(The diagram above commutes, because all the rules in $\mathcal{R}$ are in tyft-format.)

Transition system for mixed terms - II

By finality of $(\mathbf{D}^\omega, j)$ there exists a unique morphism
$\zeta : (\mathcal{S}_{\mathbf{D}^\omega}, \beta) \to (\mathbf{D}^\omega, j)$

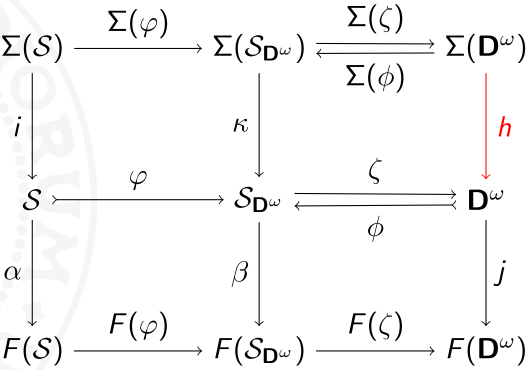
$$
\begin{array}{ccccc}
\mathcal{S} & \xrightarrow{\varphi} & \mathcal{S}_{\mathbf{D}^\omega} & \underset{\phi}{\overset{\zeta}{\rightleftarrows}} & \mathbf{D}^\omega \\
\downarrow \alpha & & \downarrow \beta & & \downarrow j \\
F(\mathcal{S}) & \xrightarrow{F(\varphi)} & F(\mathcal{S}_{\mathbf{D}^\omega}) & \xrightarrow{F(\zeta)} & F(\mathbf{D}^\omega)
\end{array}
$$

Notice that $\zeta \circ \varphi = (\!|\ |\!)$, since both $\zeta \circ \varphi$ and $(\!|\ |\!)$ are morphisms to the final F-coalgebra $(\mathbf{D}^\omega, j)$.

For the same reasons, $\zeta \circ \phi = id_{\mathbf{D}^\omega}$.

Construction of the Σ-algebra for D^ω

The following diagram resumes all the observations that have been made until now.



We define $h = \zeta \circ \kappa \circ \Sigma(\phi)$.

(This is the motivation why mixed terms have been introduced.)

Initial semantics for $\mathcal{S}$ with interpretation in $\mathbf{D}^\omega$

The initiality of $(\mathcal{S}, i)$ gives the existence of the initial semantics:

$$\begin{array}{ccc}
\Sigma(\mathcal{S}) & \xrightarrow{\Sigma([\ ])} & \Sigma(\mathbf{D}^\omega) \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} \\
\mathcal{S} & \xrightarrow{[\ ]} & \mathbf{D}^\omega
\end{array}$$

If we are able to prove that ζ is a morphism of the Σ-algebra (and not only a morphism of the F-coalgebra), we can conclude for initiality of $(\mathcal{S}, i)$ (and finality of $(\mathbf{D}^\omega, j)$) that

$$[\ ] = \zeta \circ \varphi = (\!|\ |\!)$$

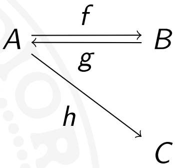
Reminder - Kernel of a function

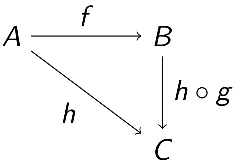
The kernel of a morphism $m : X \to Y$ is the set $K_m = \{(x, x') \in X \times X \mid m(x) = m(x')\}$.

Two lemmas

Lemma

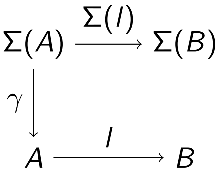
If $f \circ g = id_B$ *and* $K_f \subseteq K_h$ *then* $h \circ g \circ f = h$*.*





Lemma

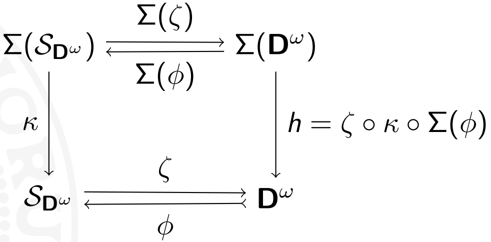
Let (A, γ) *be a* Σ*-algebra,* B *a set, and* $l : A \to B$ *a morphism*



Then K_l *is a* Σ*-congruence on* (A, γ) *iff* $K_{\Sigma(l)} \subseteq K_{l \circ \gamma}$*.*

Prove ζ is a morphism of the Σ-algebra

We prove that the following diagram commutes by applying the previous two lemmas.



1. The kernel of $K_{\Sigma(\zeta)}$ is the greatest bisimulation $\sim$ on $\mathcal{S}_{\mathbf{D}^\omega}$, the first lemma can be applied when $\sim$ is a congruence (on $(\Sigma(\mathcal{S}_{\mathbf{D}^\omega}), \kappa)$). In this case it is because our rules are in tyft-format. Then $K_{\Sigma(\zeta)} \subseteq K_{\zeta\circ\kappa}$.
2. $h = \zeta \circ \kappa \circ \Sigma(\phi)$ because of $\Sigma(\zeta) \circ \Sigma(\phi) = id_{\Sigma(\mathbf{D}^\omega)}$ (by the fact that $\zeta \circ \phi = id_{\mathbf{D}^\omega}$) and $K_{\Sigma(\zeta)} \subseteq K_{\zeta\circ\kappa}$ (point 1).

Hence, ζ is a morphism of the Σ-algebra.

Bialgebraic semantics of $\mathcal{S}$

$\mathbf{D}^\omega$ can be endowed with a structure of both an F-coalgebra and a Σ-algebra: (where $[\![\ ]\!] = \mathcal{I} \circ [\ ]$)

$$
\begin{array}{ccccc}
\Sigma(\mathcal{S}) & \xrightarrow{\Sigma([\ ])} & \Sigma(\mathbf{D}^\omega) & \xrightarrow{\Sigma(\mathcal{I})} & \Sigma(\mathbb{N}_\perp^\omega) \\
\Big\downarrow i & & \Big\downarrow h & & \Big\downarrow \iota \\
\mathcal{S} & \xrightarrow{[\ ] = (\!|\ |\!)} & \mathbf{D}^\omega & \xrightarrow{\mathcal{I}} & \mathbb{N}_\perp^\omega \\
\Big\downarrow \alpha & & \Big\downarrow j & & \\
F(\mathcal{S}) & \xrightarrow{F((\!|\ |\!))} & F(\mathbf{D}^\omega) & &
\end{array}
$$

Results

$h = \zeta \circ \kappa \circ \Sigma(\phi)$ and $[\ ] = \zeta \circ \phi = (\!|\ |\!)$

How to interpret the Σ-algebra $(\mathbf{D}^\omega, h)$ into $(\mathbb{N}^\omega_\perp, g)$

$$\begin{array}{ccc} \Sigma(\mathbf{D}^\omega) & \xrightarrow{\Sigma(\mathcal{I})} & \Sigma(\mathbb{N}^\omega_\perp) \\ \Big\downarrow h & & \Big\downarrow \iota \\ \mathbf{D}^\omega & \xrightarrow{\mathcal{I}} & \mathbb{N}^\omega_\perp \end{array}$$

$\mathcal{I}$ can be defined as (s in $\mathbf{D}^\omega$)

$$\mathcal{I}(s) = \begin{cases} \perp & \text{if } s = \surd.s' \\ \mathbf{d} & \text{if } s = \mathbf{d}.\surd.s' \\ \mathbf{d} \cdot 10^{\lceil log_{10} \mathcal{I}(\mathbf{d}'.s') \rceil} + \mathcal{I}(\mathbf{d}'.s') & \text{if } s = \mathbf{d}.\mathbf{d}'.s' \text{ and } \surd \in s' \\ \omega & \text{if } s = \mathbf{d}^\omega \end{cases}$$

Notice that the presence of ω in $\mathbb{N}^\omega_\perp$ is *crucial* to allow the above diagram commuting.