Proposition of denotational semantic for a **repeat-until** command

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Introduction

- An arithmetic expression $a \in \mathbf{A}\exp$ will denote a function $\mathrm{A}[\![a]\!]: \Sigma \to N$
- A Boolean expression $b \in \mathrm{B}\exp$ will denote a function $\mathrm{B}[\![b]\!]: \Sigma \to T,$ from the set of states to the set of truth states.
- A command *c* will denote a partial function $C[\![c]\!]: \Sigma \to \Sigma$

Denotations

- Denotations of **Aexp** and **Bexp** are intuitive and require little special attention on their own.

- Denotations of Commands on the other hand, require much more.

Denotations of Commands

- We will take the following denotations "as given" and consider them to be true
  - $C[\mathbf{skip}] = \{(\sigma, \sigma) | \sigma \in \Sigma\}$
  - $C[X := a] = \{(\sigma, \sigma[n/X]) | \sigma \in \Sigma \,\&\, n = \mathrm{A}[a]\}$
  - $C[\mathbf{if}\ b\ \mathbf{then}\ c_0\ \mathbf{else}\ c_1] =$

    $\{(\sigma, \sigma') | B[b]\sigma = \mathbf{true} \,\&\, (\sigma, \sigma') \in C[c_0]\} \bigcup$

    $\{(\sigma, \sigma') | B[b]\sigma = \mathbf{false} \,\&\, (\sigma, \sigma') \in C[c_1]\}$

Denotations of Commands …

- $C[\![\textbf{while}\ b\ \textbf{do}\ c]\!] = fix(\Gamma)$
  - Where $\Gamma(\varphi) = \{(\sigma,\sigma') \mid B[\![b]\!]\sigma \in \textbf{true} \,\&\, (\sigma,\sigma') \in \varphi \circ C[\![c]\!]\} \cup \{(\sigma,\sigma) \mid B[\![b]\!]\sigma \in \textbf{false}\}$

- What about

$$C[\![\textbf{repeat}\ c\ \textbf{until}\ b]\!] = fix(\Gamma)\ ???$$

Assignment

- In order to prove our claim, we will first prove the **repeat-until**'s natural inverse – the **while-do** loop (**repeat** *c* **until** *b* $\Leftrightarrow$ **do** *c* **while** $\neg b$)

- Therefore, by understanding and accepting the proof of **while-do**, one should have more understanding of our own definition.

While-loop

- The operational semantic of a while-loop is as follows:

$$\frac{(b,\sigma)\to \mathbf{false}}{(\mathbf{while}\ b\ \mathbf{do}\ c,\sigma)\to\sigma}$$

$$\frac{(b,\sigma)\to \mathbf{true}\quad (c,\sigma)\to\sigma''\quad (\mathbf{while}\ b\ \mathbf{do}\ c,\sigma'')\to\sigma'}{(\mathbf{while}\ b\ \mathbf{do}\ c,\sigma)\to\sigma}$$

- In order to come up with a denotional semantic, we need to note the equivalence to the **if-then-else** command.

$$w \equiv \mathbf{while}\, b\, \mathbf{do}\ c$$

$$w \sim \mathbf{if}\, b\, \mathbf{then} c; w\, \mathbf{else}\, \mathbf{skip}$$

While - continued

- So, if they are indeed equivalent (and they are) the partial function C[w] should equal that of C[**if** *b* **then** *c;w* **else skip**] by definition.
- Thus, we should have

$$C[w] = \{(\sigma,\sigma') \mid B[b]\sigma = \mathbf{true} \;\&\; (\sigma,\sigma') \in C[w] \circ C[c]\}$$
$$\bigcup \{(\sigma,\sigma) \mid B[b]\sigma = \mathbf{false}\} \qquad (*)$$

- Substituting $\varphi = C[w];\ \beta = \mathrm{B}[b];\ \gamma = C[c]$ we require a partial function φ such that

$$\varphi = \{(\sigma,\sigma') \mid \beta(\sigma) = \mathbf{true} \;\&\; (\sigma,\sigma') \in \varphi \circ \gamma\}$$
$$\bigcup \{(\sigma,\sigma) \mid B[b]\sigma = \mathbf{false}\}$$

* - a detailed proof can be found in the addendum

Problem

- As we can see, the function is “recursive” in a sense that the value we wish to know on the left side recurs on the right.

- How to solve this problem?

Problem - continued

- We can regard Γ where

$$\Gamma(\varphi) = \{(\sigma,\sigma') \mid \beta(\sigma) \in \mathbf{true} \,\&\, (\sigma,\sigma') \mid \varphi \circ \gamma\}$$
$$\bigcup \{(\sigma,\sigma) \mid \beta(\sigma) \in \mathbf{false}\}$$
$$= \{(\sigma,\sigma') \mid \exists \sigma''.\beta(\sigma) = \mathbf{true} \,\&\, (\sigma,\sigma'') \in \gamma \,\&\, (\sigma'',\sigma') \in \varphi\}$$
$$\bigcup \{(\sigma,\sigma) \mid \beta(\sigma) \in \mathbf{false}\}$$

  as a function which given φ returns $\Gamma(\varphi)$

- We want a fixed point φ of Γ so that $\varphi = \Gamma(\varphi)$

Wrapping things up

- It is not hard to check that Γ is equal to $\hat{R}$, where $\hat{R}$ is the operator on sets determined by the rule instances

$$R = \{(\{(\sigma'', \sigma')\}/(\sigma, \sigma'))/\beta(\sigma) = \mathbf{true} \,\&\, (\sigma, \sigma'') \in \gamma\}$$
$$\bigcup \{(\emptyset/(\sigma, \sigma))/\beta(\sigma) = \mathbf{false}\}$$

- And as we know(*), $\hat{R}$ has a least fixed point $\varphi = fix(\hat{R})$ where φ is a set of pairs with the property

$$\hat{R}(\theta) = \theta \Rightarrow \varphi \subseteq \theta$$

- Now that this choice for semantics agrees with the operational semantics, we can accept the statement we made earlier and define the denotational semantics for **repeat-until**.

* - a detailed proof can be found in the addendum

Repeat-until

- Similarly to **while-do**, we also have to note the equivalence with the **if-then-else** command

$$r \equiv \textbf{repeat } c \textbf{ until } b$$

$$r \sim \textbf{if } b \textbf{ then} c; \textbf{skip else } c; r$$

- By equivalence,

$$C[r] = \{(\sigma,\sigma') \mid B[b]\sigma = \textbf{false} \,\&\, (\sigma,\sigma') \in C[r] \circ C[c]\}$$

$$\bigcup \{(\sigma,\sigma) \mid B[b]\sigma = \textbf{true} \,\&\, (\sigma,\sigma') \in C[\textbf{skip}] \circ C[c]\} \qquad (*)$$

* - a detailed proof can be found in the addendum

Repeat-until…

- Substituting $\varphi = C[w];\ \beta = \mathrm{B}[b];\ \gamma = C[c];\ \varsigma = C[\mathbf{skip}]$ we get Γ where

$$\Gamma(\varphi) = \{(\sigma,\sigma') \mid \exists\sigma''.\beta(\sigma) = \mathbf{false} \,\&\, (\sigma,\sigma'') \in \gamma \,\&\, (\sigma'',\sigma') \in \varphi\}$$

$$(*)\quad \bigcup \{(\sigma,\sigma') \mid \exists\sigma''.\beta(\sigma) = \mathbf{true} \,\&\, (\sigma,\sigma'') \in \gamma \,\&\, (\sigma'',\sigma') \in \varsigma\}$$

- As shown before, $\Gamma = \hat{R}$ where

$$R = \{(\{(\sigma'',\sigma')\}/(\sigma,\sigma'))/\beta(\sigma) = \mathbf{false} \,\&\, (\sigma,\sigma'') \in \gamma\}$$

$$\bigcup \{(\{(\sigma'',\sigma')\}/(\sigma,\sigma'))/\beta(\sigma) = \mathbf{true} \,\&\, (\sigma,\sigma'') \in \varsigma\} \qquad (*)$$

$$\varphi = fix(\hat{R})$$

Q.E.D

* - a detailed proof can be found in the addendum

Operational Semantics of **repeat-until**

- As observed, the proof of denotational semantic of the **repeat-until** command doesn't differ much from that of **while-do**, as instructed by common logic.

- In the end, we will present an operational semantic of the **repeat-until** command.

$$\frac{(b,\sigma)\to \mathbf{true} \quad (c,\sigma)\to\sigma}{(\mathbf{repeat}\ c\ \mathbf{until}\ b,\sigma)\to\sigma}$$

$$\frac{(b,\sigma)\to \mathbf{false} \quad (c,\sigma)\to\sigma'' \quad (\mathbf{repeat}\ c\ \mathbf{until}\ b,\sigma'')\to\sigma'}{(\mathbf{repeat}\ c\ \mathbf{until}\ b,\sigma)\to\sigma}$$

Questions?

That’s it!

Thanks for listening.

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